The Numerical Solution of Linear Fourth Order Boundary Value Problems using Nonpolynomial Spline Technique

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Abstract: In this paper we develop a class of accurate methods based on quartic nonpolynomial spline function at midknots for the numerical solution of a fourth order two point boundary value problems associated with plate deflection theory. Using this spline function a few consistency relations are derived for computing approximations to the solution of the problem. Existing second and fourth order finite difference and spline functions based methods developed at midknots become special cases of the new approach. Convergence analysis of the proposed method is discussed. Two numerical examples are included to illustrate the practical usefulness of our method. [Journal of American Science. 2010;6(12):310-316]. (ISSN: 1545-1003).

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1. Introduction:

It is well known that the elastic beam is one of the most used elements in structures of aircrafts, buildings, ships and bridges. Beam deflection under certain load can be modeled by a fourth order two point Boundary value problems. We consider the problem of bending a rectangular simply supported beam of length L resting on an elastic foundation the vertical deflection U of the beam satisfies the system:

$$U^{(4)} + \left(Z/_D\right)U = D^{-1}q(x) \qquad (1.1)$$

$$U(0) = U(L) = U^{(2)}(0) = U^{(2)}(L) = 0 \qquad (1.2)$$

Where D is the flexural rigidity of the beam, Z is the spring constant of the elastic foundation, and the load $q(x)$ acts vertically downwards per unit length of the beam. The details of the mechanical interpretation are given in [1]. Mathematically the system (1.1) and (1.2) belongs to a general class of boundary value problems of the form

$$y^{(4)} + f(x)\,y = g(x) \quad , \quad x \in [a,b] \qquad (1.3)$$

Subject to the boundary conditions

$$y(a) = A_1 \ , y(b) = A_2 \, , y^{(2)}(a) = B_1 \, , y^{(2)}(b) = B_2 \qquad (1.4)$$

Where $f(x)$ and $g(x)$ are continuous on $[a,b]$ and $A_{i\,,}B_i$ $(i = 1,2)$ are finite real arbitrary constants. The analytical solution of (1.3) subject to (1.4) cannot be obtained for arbitrary choices of $f(x)$ and $g(x)$. The numerical analysis literature contains other methods developed to find an approximate solution of this problem using spline functions and finite difference.

Usmani [2], Usmani and Warsi [3] solved linear fourth order two point boundary value problems using quartic, quinitic and sextic polynomial spline functions. Al-Said et al. [4,5] solved fourth order obstacle problems using cubic and quartic spline functions, respectively. Usmani [6] solved this problem with the boundary conditions involving first derivatives using quintic and sextic polynomial spline functions. Also, Rashidinia and Golbabaee [7] and Siddiqi and Ghazala [8] solved the preceding problem using quintic spline functions. VanDaele et al. [9] solved the above boundary value problem with the boundary conditions involving first derivatives using nonpolynomial spline function. Zhu [10] introduced optimal quartic spline collocation methods for the numerical solution of this problem based on perturbation technique which gives rise to two optimal quartic spline one step and three step collocation methods.

Al-Said et al. [11] developed a fourth order finite difference method for the system (1.3) and (1.4). Ramadan et al. [12, 13] solved this problem using quintic nonpolynomial spline function. The aim of this paper is to construct a new spline method based on a nonpolynomial spline function that has a polynomial part and a trigonometric part to develop numerical methods for obtaining smooth approximations for the solution of the system (1.3) and (1.4); the paper is organized as follows: in section 2, we present the derivation of our method. The method is formulated in a matrix form in section 3. Convergence analysis for second, fourth and six order methods is established in section 4. Numerical results are presented to illustrate the applicability and accuracy in section 5. Finally, in section 6, the results of the proposed methods are concluded to illustrate their practical usefulness and accuracy.

2. Derivation of the method:

We introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into n equal parts.

$x_i = a + ih$, $i = 0,1,\dots\dots,n$

$$x_0 = a \ \ , x_n = b \ and \ h = \frac{b-a}{n} \qquad (2.1)$$

Let $y(x)$ be the exact solution of the system (1.3) and (1.4) and s_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}).

Each nonpolynomial spline segment $Q_i(x)$ has the form:

$$Q_i(x) = a_i \sin k\,(x-x_i) + b_i \cos k\,(x-x_i) + c_i\,(x-x_i)^2 + d_i\,(x-x_i) + e_i \qquad (2.2)$$

$i = 0,1,2,\dots\dots,n-1$,where a_i ,b_i ,c_i d_i, and e_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and Eq. (2.2) reduces to quartic polynomial spline function in $[a,b]$ when $k \to 0$ choosing the spline function in this form will enable us to generalize other existing methods by arbitrary choices of the parameters α , $\beta$and γ which will be defined later in the end of this section. Thus, our quartic nonpolynomial spline is now defined by the relations:

(i) $S(x) = Q_i(x)$, $x \in [x_i, x_{i+1}], i = 0,1,\dots n-1$

$$(ii)\ S(x) \in C^{\infty}[a,b] \qquad (2.3)$$

First, we develop expressions for the five coefficients of (2.2) in terms of $S_{i+1/2}, D_i, M_{i+1/2}$ and $T_i, W_{i+1/2}$, where

$$(i)\ Q_i\left(x_{i+1/2}\right) = S_{i+1/2}, Q_i^{(1)}(x_i) = D_i$$
$$(ii)\ Q_i^{(2)}\left(x_{i+1/2}\right) = M_{i+1/2} \qquad (2.4)$$
$$(iii)\ Q_i^{(3)}(x_i) = T_i \ \ , \ \ Q_i^{(4)}\left(x_{i+1/2}\right) = W_{i+1/2}$$

We obtain via a straight forward calculation the following expressions:

$$a_i = \frac{-T_i}{k^3} \ \ ; \quad b_i = \frac{T_i}{k^3}\tan(\theta/2) + \frac{1}{k^4\cos(\theta/2)} W_{i+1/2}$$
$$c_i = \frac{1}{2} M_{i+1/2} + \frac{1}{2k^2} W_{i+1/2} \ \ ; \ \ d_i = D_i + \frac{T_i}{k^3} \qquad (2.5)$$
$$e_i = S_{i+1/2} - \frac{h}{2}D_i - \frac{h^2}{8}M_{i+1/2} - \frac{h}{2k^2}T_i - \left[\frac{1}{k^4} + \frac{h^2}{8k^2}\right] W_{i+1/2}$$

Where $\theta = kh$ and $i = 0,1,2,\dots\dots,n-1$ Now using the continuity (ii) in (2.3) that is the continuity of quartic nonpolynomial spline $S(x)$ and its derivatives up to order three are involved at the point (x_i,s_i) where the two quartics $Q_{i-1}(x)$ and $Q_i(x)$ join. Thus, $Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i)$, $m = 0,1,2$ and 3 which on using Eqs. (2.4) and (2.5) yield the following consistency relations:

$$\frac{h}{2}[D_i + D_{i-1}] = \left(S_{i+1/2} - S_{i-1/2}\right) - \frac{h^2}{8}\left[M_{i+1/2} + 3M_{i-1/2}\right] + \left[\frac{\tan\theta/2}{k^3} - \frac{h}{2k^2}\right][T_i + T_{i-1}] + \left[\frac{1}{k^4\cos(\theta/2)} - \frac{1}{k^4} - \frac{h^2}{8k^2}\right] W_{i+1/2} + \left[\frac{1}{k^4} - \frac{\cos(\theta)}{k^4\cos(\theta/2)} - \frac{3h^2}{8k^2}\right] W_{i-1/2} \qquad (2.6)$$

$$\frac{h}{2}[D_i - D_{i-1}] = \frac{h^2}{2} M_{i-1/2} + \left[\frac{h^2}{2k^2} - \frac{h\sin(\theta/2)}{k^3}\right] W_{i-1/2} \qquad (2.7)$$

$$\frac{\tan(\theta/2)}{k}[T_i + T_{i-1}] = \left[M_{i+1/2} - M_{i-1/2}\right] + \left[\frac{-1+\cos(\theta/2)}{k^2\cos(\theta/2)}\right] W_{i+1/2} + \left[\frac{\cos\theta - \cos(\theta/2)}{k^2\cos(\theta/2)}\right] W_{i-1/2} \qquad (2.8)$$

$$\frac{\tan(\theta/2)}{k}[T_i - T_{i-1}] = \frac{2\sin^2(\theta/2)}{k^2\cos(\theta/2)} W_{i-1/2} \qquad (2.9)$$

Adding Eqs. (2.6) and (2.7) then use equation (2.8), it follows that

$$hD_i = \left(S_{i+1/2} - S_{i-1/2}\right) + \left[\frac{1}{k^2} - \frac{h}{2k\tan(\theta/2)} - \frac{h^2}{8}\right]\left(M_{i+1/2} - M_{i-1/2}\right) + \left[\frac{h^2}{8k^2} + \frac{h}{2k^3\tan(\theta/2)} - \frac{h}{2k^3\sin(\theta/2)}\right]\left(W_{i-1/2} - W_{i+1/2}\right) \qquad (2.10)$$

Adding Eqs. (2.8) and (2.9), it follows that

$$T_i = \frac{k}{2\tan(\theta/2)}\left(M_{i+1/2} - M_{i-1/2}\right) + \left(\frac{1}{2k\sin\theta/2} - \frac{1}{2k\tan\theta/2}\right)\left(W_{i-1/2} - W_{i+1/2}\right) \qquad (2.11)$$

Eliminating $T's$ from Eqs. (2.9) and (2.11), it follows that:

$$\left(\frac{1}{2k\tan\theta/2} - \frac{1}{2k\sin\theta/2}\right)\left(W_{i-3/2} + W_{i+1/2}\right) + \left(\frac{1}{k\sin\theta/2} - \frac{1}{k\tan\theta/2} - \frac{2\sin\theta/2}{k}\right) W_{i-1/2} = \left(\frac{k}{2\tan\theta/2}\right)\left(-M_{i+1/2} + 2M_{i-1/2} - M_{i-3/2}\right) \qquad (2.12)$$

Eliminating $D's$ from Eqs. (2.7) and (2.10) then use equation (2.12) it follows that:

$$h^2 M_{i-1/2} = \left(S_{i-3/2} - 2S_{i-1/2} + S_{i+1/2}\right) + \left(\frac{1}{k^4 cos\theta/2} - \frac{h^2}{8k^2 cos\theta/2} - \frac{1}{k^4}\right)\left(W_{i-3/2} + W_{i+1/2}\right)$$

$$+\left(\frac{2+4\tan\theta/2 sin\,\theta/2}{k^4} + \frac{h^2}{4k^2 cos\theta/2} - \frac{h^2 sin\theta/2\, tan\theta/2}{2k^2} - \frac{2}{k^4 cos\theta/2} - \frac{h^2}{k^2}\right) W_{i-1/2} \qquad (2.13)$$

Eliminating $M's$ from the Eqs. (2.12) and (2.13), it follows that:

$$\begin{aligned} S_{i-5/2} - 4S_{i-3/2} + 6S_{i-1/2} - 4S_{i+1/2} + S_{i+3/2} &= h^4\Big[\alpha\left(W_{i-5/2} + W_{i+3/2}\right) \\ &+ \beta\left(W_{i-3/2} + W_{i+1/2}\right) \\ &+ \gamma W_{i-1/2}\Big], i = 3,.,n-2 \end{aligned} \qquad (2.14)$$

Where $W_i = f_i S_i + g_i$, with $f_i = f(x_i)$ and $g_i = g(x_i)$,

$$\alpha = \frac{-8\,tan\,\theta/2 + \theta^2 tan\,\theta/2 + 8\,sin\,\theta/2}{8\,\theta^4\,sin\,\theta/2}$$

$$\beta = \frac{8 + 3\,\theta^2 - 16\,cos\,\theta/2 + (8-\theta^2)\cos\theta}{4\,\theta^4\,cos\,\theta/2}$$

$$\gamma = \frac{-8 + \theta^2 + 24\,cos(\theta/2) + (16 + 6\,\theta^2)\cos\theta}{4\,\theta^4\,cos\,\theta/2}$$

If $k \to 0$ that is $\theta \to 0, (\alpha, \beta, \gamma) \to \left(\frac{1}{384}, \frac{76}{384}, \frac{230}{384}\right)$ So that the relation (2.14) reduce to quartic polynomial spline relation [2].

Eq. (2.14) gives (n-4) linear algebraic equations in the (n) unknowns $S_{i+1/2}, i = 0,1,2,\ldots\ldots,n-1$, so we need four more equations, two at each end of the range of integration for direct computation of $S_{i+1/2}$. These four equations are deduced by Taylor series along with the method of undetermined coefficients.

$$10S_{1/2} - 5\,S_{3/2} + S_{5/2} = 6\,S_0 - \frac{5}{4}\,h^2 S_0^{(2)} + h^4\left[\alpha_0 S_0^{(4)} + \sum_{J=1}^{5}\alpha_J S_{J-(1/2)}^{(4)}\right], for\ i = 1 \qquad (2.15)$$

$$-5S_{1/2} + 6\,S_{3/2} - 4S_{5/2} + S_{7/2} = -2\,S_0 - \frac{h^2}{4}\,S_0^{(2)} + h^4\left[\sum_{J=1}^{6}\beta_J S_{J-(1/2)}^{(4)}\right], for\ i = 2 \qquad (2.16)$$

$$S_{n-7/2} - 4\,S_{n-5/2} + 6S_{n-3/2} - 5S_{n-1/2} = -2\,S_n - \frac{h^2}{4}\,S_n^{(2)} + h^4\left[\sum_{J=1}^{6}\beta_{7-J} S_{n+J-(13/2)}^{(4)}\right], for\ i = n-1 \qquad (2.17)$$

$$S_{n-5/2} - 5\,S_{n-3/2} + 10S_{n-1/2} = 6S_n - \frac{5}{4}\,h^2 S_n^{(2)} + h^4\left[\alpha_0 S_n^{(4)} + \sum_{J=1}^{5}\alpha_{6-J} S_{n+J-(11/2)}^{(4)}\right], for\ i = n \qquad (2.18)$$

The local truncation errors $t_i, i = 1,2,\ldots\ldots,n$ associated with the scheme (2.14 – 2.18) can be obtained as follows: first we rewrite the scheme (2.14 – 2.18) in the form:

$$10y_{1/2} - 5\,y_{3/2} + y_{5/2} = 6\,y_0 - \frac{5}{4}\,h^2 y_0^{(2)} + h^4\left[\alpha_0 y_0^{(4)} + \sum_{J=1}^{5}\alpha_J y_{J-(1/2)}^{(4)}\right] + t_1;\ at\ i = 1 \qquad (2.19)$$

$$-5y_{1/2} + 6\,y_{3/2} - 4y_{5/2} + y_{7/2} = -2\,y_0 - \frac{h^2}{4}\,y_0^{(2)} + h^4\left[\sum_{J=1}^{6}\beta_J y_{J-(1/2)}^{(4)}\right] + t_2;\ at\ i = 2 \qquad (2.20)$$

$$\begin{aligned} y_{i-5/2} - 4y_{i-3/2} + 6y_{i-1/2} - 4y_{i+1/2} + y_{i+3/2} &= \Big[\alpha\left(y_{i-5/2}^{(4)} + y_{i+3/2}^{(4)}\right) \\ &+ \beta\left(y_{i-3/2}^{(4)} + y_{i+1/2}^{(4)}\right) \\ &+ \gamma\,y_{i-1/2}^{(4)}\Big] + t_i; \\ &at\ i = 3,4,\ldots\ldots,n-2 \end{aligned} \qquad (2.21)$$

$$y_{n-7/2} - 4\,y_{n-5/2} + 6y_{n-3/2} - 5y_{n-1/2} = -2\,y_n - \frac{h^2}{4}\,y_n^{(2)} + h^4\left[\sum_{J=1}^{6}\beta_{7-J}\,y_{n+J-(13/2)}^{(4)}\right] + t_{n-1};\ at\ i = n-1 \qquad (2.22)$$

$$y_{n-5/2} - 5\,y_{n-3/2} + 10y_{n-1/2} = 6y_n - \frac{5}{4}\,h^2 y_n^{(2)} + h^4[\alpha_0 y_n^{(4)} + \sum_{J=1}^{5}\alpha_{6-J}\,y_{n+J-(11/2)}^{(4)}\ ] + t_n;\ at\ i = n \qquad (2.23)$$

The terms $y_{i-1/2}\ and\ y_{i-1/2}^{(4)}, \ldots\ldots\ldots$ in Eq. (2.21) are expanded around the point x_i using Taylor series and the expressions for $t_i, i = 3,4,\ldots,n-2$ can be obtained. Also, expressions for $t_i, i = 1,2,n-1,n$ are obtained in a similar manner by expanding around $x_0, for\ i = 1,2$ and around $x_n, for\ i = n-1,n$, The local truncation errors $t_i, i = 3,4,\ldots,n-2$ associated with the scheme (2.14) are

$$t_i = \left(1-(2\alpha+2\beta+\gamma)\right) h^4\, y_i^{(4)} + \frac{1}{2}(-1+2\alpha+2\beta+\gamma)h^5\, y_i^{(5)} + \frac{1}{24}\left(7-3(34\alpha+10\beta+\gamma)\right) h^6\, y_i^{(6)} + \frac{1}{48}(-5+98\alpha+26\beta+\gamma)h^7 y_i^{(7)} + \frac{1}{1920}\left(69-5(706\alpha+82\beta+\gamma)\right)h^8 y_i^{(8)} + \frac{1}{11520}(-115+84646\alpha+726\beta+3\gamma)\, h^9 y_i^{(9)} + \frac{1}{967680}\left(2497-21(16354\alpha+730\beta + \gamma)\right) h^{10}\, y_i^{(10)} + O(h^{11}),$$
$$at\ i = 3,.,n-2 \tag{2.24}$$

The scheme (2.14 – 2.18) gives rise to a class of methods of different orders as follows:

(I) Second order method

For any choice of arbitrary α and β with $\gamma = 1-2(\alpha+\beta)$

And $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \left(\frac{-77}{192}, \frac{192}{192}, 0,0,0,0\right)$

$(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6) = \left(\frac{-3}{384}, \frac{385}{384}, 0,0,0,0\right)$

Then the local truncation errors for $(\alpha,\beta,\gamma) = \left(\frac{3}{200}, \frac{30}{200}, \frac{134}{200}\right)$ are

$$t_i = \begin{cases} \frac{623}{4608}\, h^6\, y_i^{(6)} + O(h^7) \ ;\ i = 1,n \\ \frac{1897}{11520}\, h^6\, y_i^{(6)} + O(h^7) \ ;\ i = 2,n-1 \\ \frac{-13}{300}\, h^6\, y_i^{(6)} + O(h^7) \ ;\ i = 3,\dots\dots,n-2 \end{cases} \tag{2.25}$$

(II) Fourth order method

For any choice of arbitrary α with $\beta = \frac{1-24\alpha}{6}$ and $\gamma = 1-2(\alpha+\beta)$

And

$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \left(\frac{-108}{5760}, \frac{2410}{5760}, \frac{1195}{5760}, \frac{-47}{5760}, 0, 0\right)$

$(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6) = \left(\frac{3575}{23040}, \frac{15629}{23040}, \frac{3677}{23040}, \frac{39}{23040}, 0, 0\right)$

Then the local truncation errors for $(\alpha,\beta,\gamma) = \left(\frac{-7}{4850}, \frac{2509}{14550}, \frac{4787}{7275}\right)$ are

$$t_i = \begin{cases} \frac{9587}{3096576}\, h^8\, y_i^{(8)} + O(h^9) \ ;\ i = 1,n \\ \frac{-44761}{15482880}\, h^8\, y_i^{(8)} + O(h^9) \ ;\ i = 2,n-1 \\ \frac{19}{349200}\, h^8\, y_i^{(8)} + O(h^9) \ ;\ i = 3,\dots\dots,n-2 \end{cases} \tag{2.26}$$

(III) Six order method

For $(\alpha,\beta,\gamma) = \left(\frac{-1}{720}, \frac{124}{720}, \frac{474}{720}\right)$ and

$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=$

$\left(\frac{-622}{589935}, \frac{856}{2265}, \frac{358}{1416}, \frac{-790}{18878}, \frac{354}{28317}, \frac{-395}{226534}\right)$

$(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6) =$

$\left(\frac{108}{719}, \frac{253}{360}, \frac{458}{3995}, \frac{661}{239720}, \frac{-120}{7192}, \frac{661}{239720}\right)$

Then the local truncation errors are

$$t_i = \begin{cases} \frac{833}{902891}\, h^{10}\, y_i^{(10)} + O(h^{11}) \ ;\ i = 1,n \\ \frac{-3307}{1538523}\, h^{10}\, y_i^{(10)} + O(h^{11}) \ ;\ i = 2,n-1 \\ \frac{1}{3024}\, h^{10}\, y_i^{(10)} + O(h^{11}) \ ;\ i = 3,\dots\dots,n-2 \end{cases} \tag{2.27}$$

Remark:

(i) When $\alpha = \frac{1}{384}, \beta = \frac{76}{384}$ and $\gamma = \frac{230}{384}$ then the scheme (2.14) is reduced to Usmani method based on quartic polynomial spline [2].

(ii) When $\alpha = 0, \beta = \frac{62526}{375156}$ and $\gamma = \frac{250104}{375156}$ then the scheme (2.14) is reduced to Al-Said and Noor based on finite difference method [11].

(iii) When $\alpha = 0, \beta = \frac{1}{24}$ and $\gamma = \frac{22}{24}$ then the scheme (2.14) is reduced to Al-Said and Noor based on cubic polynomial spline method [5].

3. Spline solutions:

The spline solution of (1.3) with the boundary condition (1.4) is based on the linear equations given by (2.14 – 2.18) , Let $Y = \left(y_{i+1/2}\right), S = \left(s_{i+1/2}\right), C = (c_i), T = (t_i), E = e_{i+1/2} = Y - S$ be n-dimensional column vectors, then we can write the standard matrix equations in the form:

$$(i) NY = C + T$$
$$(ii) NS = C \tag{3.1}$$
$$(iii) NE = T$$

We also have $N = N_0 + h^4 BF, F = diag\left(f_{i+1/2}\right)$ (3.2)

Where

$$N_0 = \begin{bmatrix} 10 & -5 & 1 & & & & & \\ -5 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & & & & & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -5 \\ & & & & 1 & -5 & 10 \end{bmatrix}$$

(3.3)

The matrix B has the form

$$B = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & & \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \\ \alpha & \beta & \gamma & \beta & \alpha & & \\ & & & & & & \\ & & \alpha & \beta & \gamma & \beta & \alpha \\ & \beta_6 & \beta_5 & \beta_4 & \beta_3 & \beta_2 & \beta_1 \\ & & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}$$

(3.4)

For the vector C, we have

$$C_i = \begin{cases} 6A_1 - \frac{5}{4}h^2\beta_1 + h^4\left[\alpha_0\left(-A_1 f_0 + g_0\right) + \sum_{J=1}^{5}\alpha_J g_{J-(1/2)}\right] & , i=1 \\ -2A_1 - \frac{h^2}{4}\beta_1 + h^4\left[\sum_{J=1}^{6}\beta_J g_{J-(1/2)}\right] & , i=2 \\ h^4\left[\alpha\left(g_{i-5/2} + g_{i+3/2}\right) + \beta\left(g_{i-3/2} + g_{i+1/2}\right) + \gamma\left(g_{i-1/2}\right)\right], & i=3,\ldots\ldots,n-2 \\ -2A_2 - \frac{h^2}{4}\beta_2 + h^4\left[\sum_{J=1}^{6}\beta_{7-J}\, g_{n+J-(13/2)}\right] & , i=n-1 \\ 6A_2 - \frac{5}{4}h^2\beta_2 + h^4\left[\alpha_0\left(-A_2 f_n + g_n\right) + \sum_{J=1}^{5}\alpha_J g_{n+J-(11/2)}\right] & , i=n \end{cases}$$

(3.5)

4. Convergence analysis

Our main purpose now is to derive a bound on $\|E\|$ where $\| \ . \ \|$ represents the infinity norm. Now we turn back to the error equation (iii) in (3.1) and rewrite it in the form

$$E = N^{-1}T = (N_0 + h^4 BF)^{-1}T$$
$$= (I - N_0^{-1}h^4 BF)^{-1}\, N_0^{-1}\, T$$

We get

$$\|E\| \le \frac{\|N_0^{-1}\| \ \|T\|}{1 - h^4\|N_0^{-1}\| \ \|B\| \ \|F\|} \qquad (4.1)$$

Provided that $h^4\|N_0^{-1}\| \ \|B\| \ \|F\| < 1$

It was shown in [2] that N_0 is nonsingular and its inverse satisfies the inequality

$$\|N_0^{-1}\| \le \frac{5(b-a)^4 + 10(b-a)^2\, h^2 + 9h^4}{384\, h^4} = O(h^{-4}) \qquad (4.2)$$

Lemma 4.1. The matrix N given by (3.2) is nonsingular, provided that $\|B\|\, |f(x)|\, \widetilde{w} < 1$ (4.3)

Where $\widetilde{w} = \frac{5(b-a)^4 + 10(b-a)^2\, h^2 + 9h^4}{384}$ and $\|B\|$ is a finite number; the proof of this lemma follows from the following statement [14]; if G is a square matrix of order n and $\|G\| < 1$ then $(I+G)^{-1}$ exists and $\|(I+G)^{-1}\| < \frac{1}{1-\|G\|}$

As a consequence of lemma 4.1, the discrete boundary value problem has a unique solution if $\|B\|\, |f(x)|\, \widetilde{w} < 1$.

Now from Eqs. (2.25 – 2.27) we investigate the following three cases:

Case (*i*): second order convergent method

We have from Eq (2.25)

$$\|T\| = \frac{1897}{11520}\, h^6\, M_6, M_6 = max_{a \le x \le b} \left|y^{(6)}(x)\right| \qquad (4.4)$$

And then it follows that:

$$\|E\| \le \frac{1897\widetilde{w} M_6\, h^2}{11520[1 - \widetilde{w}\, \|B\|\, |f(x)|]} = G_2\, h^2 \ \cong O\,(h^2) \qquad (4.5)$$

Where $G_2 = \frac{1897\widetilde{w} M_6}{11520[1 - \widetilde{w}\, \|B\|\, |f(x)|]}$

Case (*ii*) : fourth order convergent method

We have from Eq (2.26)

$$\|T\| = \frac{9587}{3096576}\, h^8\, M_8, M_8 = max_{a \le x \le b} \left|y^{(8)}(x)\right| \qquad (4.6)$$

And then it follows that:

$$\|E\| \le \frac{9587\, \widetilde{w} M_8\, h^4}{3096576[1 - \widetilde{w}\, \|B\|\, |f(x)|]} = G_4\, h^4 \ \cong O\,(h^4) \qquad (4.7)$$

Where $G_4 = \frac{9587\, \widetilde{w}\, M_8}{3096576[1 - \widetilde{w}\, \|B\|\, |f(x)|]}$

Case (*iii*): six order convergent method

We have from Eq (2.27)

$$\|T\| = \frac{3307}{1538523}\, h^{10}\, M_{10}, M_{10} = max_{a \le x \le b} \left|y^{(10)}(x)\right| \qquad (4.8)$$

And then it follows that:

$$\|E\| \le \frac{3307\, \widetilde{w} M_{10}\, h^6}{1538523[1 - \widetilde{w}\, \|B\|\, |f(x)|]} = G_6\, h^6 \ \cong O\,(h^6) \qquad (4.9)$$

Where $G_6 = \frac{3307\, \widetilde{w}\, M_{10}}{1538523[1 - \widetilde{w}\, \|B\|\, |f(x)|]}$

We summarize the above results in the next theorem

Theorem 4.1

Let $y(x)$ be the exact solution of the continuous boundary value problem (1.3) with the boundary condition (1.4) and let $y_{i+1/2}, i = 0, 1, \dots, n-1$, satisfy the discrete boundary value problem (*ii*) in (3.1). Further, if $e_{i+1/2} = y_{i+1/2} - S_{i+1/2}$ then

(1) $\|E\| \cong O\ (h^2)$, is a second order method which is given by (4.5).

(2) $\|E\| \cong O\ (h^4)$, is a fourth order method which is given by (4.7).

(3) $\|E\| \cong O\ (h^6)$, is a six order method which is given by (4.9).

5. Numerical examples and discussion:

We now consider two numerical examples to illustrate the comparative performance of our method (*ii*) in (3.1) over other existing methods. All calculations are implemented by MATLAB 7 .

Example 1. Consider the boundary value problem

$$y^{(4)} - y = -4(2x\cos(x) + 3\sin(x)) \qquad (5.1)$$

$$y(0) = y(1) = 0, y^{(2)}(0) = 0, y^{(2)}(1) = 2\,sin(1) + 4\cos(1)$$

The analytical solution of (5.1) is

$$y(x) = (x^2 - 1)\sin x \qquad (5.2)$$

The numerical results for our second, fourth and six orders are summarized in Table 1.

*Table1: The observed maximum errors for Example*1

h	Six order method	Fourth order method	Second order method
$\frac{1}{8}$	5.07 – 10	3.77 – 8	5.10- 5 [a]
$\frac{1}{16}$	7.81 – 12	3.18 – 10	2.97 – 5
$\frac{1}{32}$	1.02 – 13	9.11 – 12	9.92 – 6

[a] 5.10 – 5 = 5.10×10^{-5}

Example 2. Consider the boundary value problem

$$y^{(4)} + xy = -(8 + 7x + x^3)e^x \qquad (5.3)$$

$$y(0) = y(1) = 0, y^{(2)}(0) = 0, y^{(2)}(1) = -4e$$

The analytical solution of (5.3) is

$$y(x) = x(1-x)e^x \qquad (5.4)$$

Table 2: The observed maximum errors for example 2

h	Six order method	Fourth order method	Second order method
$\frac{1}{8}$	1.48 – 9	1.08 – 7	1.91 – 4
$\frac{1}{16}$	2.22 – 11	1.13 – 9	6.98 – 5
$\frac{1}{32}$	5.79 – 13	2.92 – 11	2.54 – 5

Table 3: The observed maximum errors for example 2

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Our six order method	1.48 – 9	2.22 – 11	5.79 – 13
Ramadan et al. [13]	1.76 – 8	2.98 – 10	4.75 – 12
Ramadan et al. [12]	1.91 – 7	3.12 – 9	5.02 – 11
Al-Said and Noor [11]	2.86 – 7	2.27 – 8	1.49 – 9
Zhu [10]	6.90 – 7	1.30 – 8	2.20 – 10
Usmani and Warsi [3]	1.26 – 6	7.87 – 8	4.91 – 9
Al-Said et al. [5]	5.69 – 4	1.47 – 4	3.71 – 5
Usmani [2]	4.24 – 4	1.08 – 4	2.70 – 5
Usmani and Warsi [3]	8.67 – 4	2.16 – 4	5.40 – 5
Al-Said and Noor [4]	1.62 – 3	6.39 – 4	5.88 – 5

The numerical results for our six, fourth and second ordered methods are summarized in Tables 1-3 and compared with other existing methods. The results in Table 3 clearly show superiority over the existing methods and also confirm that on halving the step size h , the $\|E\|$ is approximately reduced by a factor of $\frac{1}{2^p}$ where p is the order of the numerical method.The proposed quartic nonpolynomial spline method generalizes other existing methods through arbitrary choices of α, β and γ where we get six, fourth and second ordered methods.

6. Conclusion:

Three new methods are presented for solving fourth order two point boundary value problem using quartic nonpolynomial spline. These methods are shown to be optimal second, fourth and six ordered methods which are better than other existing methods [2-5, 10-13]. Introduction of the parameter k in the trigonometric part of the nonpolynomial spline function of the present methods improves the accuracy of the schemes which is evident from the numerical results given in tables 1-3, and these results show that the proposed methods maintain a very remarkable high accuracy which make them are very encouraging for dealing with the solution of two point boundary value problems.

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